1. A curve has the equation

$$2x^2 + xy - y^2 + 18 = 0.$$

Find the coordinates of the points where the tangent to the curve is parallel to the $x$-axis. **(8)**

---

2. Use the substitution $x = 2\tan u$ to show that

$$\int_0^2 \frac{x^2}{x^2+4}\,\mathrm{d}x = \tfrac{1}{2}(4 - \pi). \qquad \textbf{(8)}$$

---

3. *(a)* Show that $(1\tfrac{1}{24})^{-\frac{1}{2}} = k\sqrt{6}$, where $k$ is rational. **(2)**

*(b)* Expand $(1 + \tfrac{1}{2}x)^{-\frac{1}{2}}$, $|x| < 2$, in ascending powers of $x$ up to and including the term in $x^3$, simplifying each coefficient. **(4)**

*(c)* Use your answer to part *(b)* with $x = \tfrac{1}{12}$ to find an approximate value for $\sqrt{6}$, giving your answer to 5 decimal places. **(3)**

---

4. Relative to a fixed origin, two lines have the equations

$$\mathbf{r} = (7\mathbf{j} - 4\mathbf{k}) + s(4\mathbf{i} - 3\mathbf{j} + \mathbf{k}),$$

and

$$\mathbf{r} = (-7\mathbf{i} + \mathbf{j} + 8\mathbf{k}) + t(-3\mathbf{i} + 2\mathbf{k}),$$

where $s$ and $t$ are scalar parameters.

*(a)* Show that the two lines intersect and find the position vector of the point where they meet. **(5)**

*(b)* Find, in degrees to 1 decimal place, the acute angle between the lines. **(4)**

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**5.** A curve has parametric equations

$$x = \frac{t}{2-t}, \quad y = \frac{1}{1+t}, \quad -1 < t < 2.$$

*(a)* Show that $\dfrac{dy}{dx} = -\tfrac{1}{2}\left(\dfrac{2-t}{1+t}\right)^2$. **(4)**

*(b)* Find an equation for the normal to the curve at the point where $t = 1$. **(3)**

*(c)* Show that the cartesian equation of the curve can be written in the form

$$y = \frac{1+x}{1+3x}.$$ **(4)**

---

**6.** *(a)* Find $\int \tan^2 x \, dx$. **(3)**

*(b)* Show that

$$\int \tan x \, dx = \ln|\sec x| + c,$$

where $c$ is an arbitrary constant. **(4)**

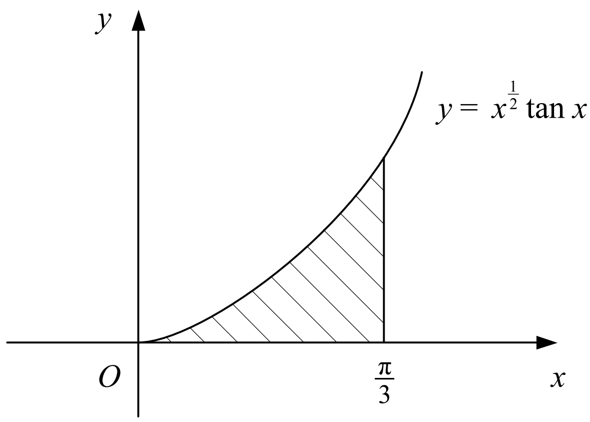


**Figure 1**

Figure 1 shows part of the curve with equation $y = x^{\frac{1}{2}} \tan x$.

The shaded region bounded by the curve, the $x$-axis and the line $x = \frac{\pi}{3}$ is rotated through $2\pi$ radians about the $x$-axis.

*(c)* Show that the volume of the solid formed is $\frac{1}{18}\pi^2(6\sqrt{3} - \pi) - \pi \ln 2$. **(6)**

---

***Turn over***

**7.**

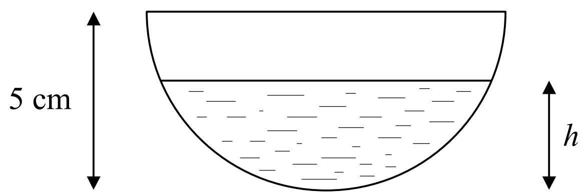


**Figure 2**

Figure 2 shows a hemispherical bowl of radius 5 cm.

The bowl is filled with water but the water leaks from a hole at the base of the bowl. At time $t$ minutes, the depth of water is $h$ cm and the volume of water in the bowl is $V$ cm$^3$, where

$$V = \tfrac{1}{3}\pi h^2(15 - h).$$

In a model it is assumed that the rate at which the volume of water in the bowl decreases is proportional to $V$.

*(a)* Show that

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{kh(15-h)}{3(10-h)},$$

where $k$ is a positive constant. **(5)**

*(b)* Express $\dfrac{3(10-h)}{h(15-h)}$ in partial fractions. **(3)**

Given that when $t = 0$, $h = 5$,

*(c)* show that

$$h^2(15 - h) = 250\,\mathrm{e}^{-kt}.$$ **(6)**

Given also that when $t = 2$, $h = 4$,

*(d)* find the value of $k$ to 3 significant figures. **(3)**

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**END**

# C4 Paper F – Marking Guide

| | | | | |
|---|---|---|---|---|
| **1.** | $4x + y + x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$ | | M1 A2 | |
| | SP: $\frac{dy}{dx} = 0 \quad \therefore\ 4x + y = 0, \quad y = -4x$ | | M1 A1 | |
| | sub. $2x^2 - 4x^2 - 16x^2 + 18 = 0$ | | M1 | |
| | $x^2 = 1,\ x = \pm 1 \quad \therefore\ (-1, 4), (1, -4)$ | | A2 | **(8)** |
| **2.** | $x = 2\tan u \ \Rightarrow\ \frac{dx}{du} = 2\sec^2 u$ | | M1 | |
| | $x = 0 \ \Rightarrow\ u = 0,\ x = 2 \ \Rightarrow\ u = \frac{\pi}{4}$ | | B1 | |
| | $I = \int_0^{\frac{\pi}{4}} \frac{4\tan^2 u}{4\sec^2 u} \times 2\sec^2 u \ du = \int_0^{\frac{\pi}{4}} 2\tan^2 u \ du$ | | A1 | |
| | $= \int_0^{\frac{\pi}{4}} (2\sec^2 u - 2)\ du$ | | M1 | |
| | $= [2\tan u - 2u]_0^{\frac{\pi}{4}}$ | | M1 A1 | |
| | $= (2 - \frac{\pi}{2}) - (0) = \frac{1}{2}(4 - \pi)$ | | M1 A1 | **(8)** |
| **3.** | *(a)* | $= (\frac{25}{24})^{-\frac{1}{2}} = \sqrt{\frac{24}{25}} = \sqrt{\frac{4}{25}\times 6} = \frac{2}{5}\sqrt{6} \quad [\ k = \frac{2}{5}\ ]$ | M1 A1 | |
| | *(b)* | $= 1 + (-\frac{1}{2})(\frac{1}{2}x) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}(\frac{1}{2}x)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3\times 2}(\frac{1}{2}x)^3 + \ldots$ | M1 | |
| | | $= 1 - \frac{1}{4}x + \frac{3}{32}x^2 - \frac{5}{128}x^3 + \ldots$ | A3 | |
| | *(c)* | $x = \frac{1}{12} \ \Rightarrow\ (1 + \frac{1}{2}x)^{-\frac{1}{2}} = (1\frac{1}{24})^{-\frac{1}{2}} = \frac{2}{5}\sqrt{6}$ | | |
| | | $x = \frac{1}{12} \ \Rightarrow\ (1 + \frac{1}{2}x)^{-\frac{1}{2}} \approx 1 - \frac{1}{4}(\frac{1}{12}) + \frac{3}{32}(\frac{1}{12})^2 - \frac{5}{128}(\frac{1}{12})^3$ | M1 | |
| | | $= 0.97979510$ | | |
| | | $\therefore\ \sqrt{6} \approx \frac{5}{2} \times 0.97979510 = 2.44949$ (5dp) | M1 A1 | **(9)** |
| **4.** | *(a)* | $4s = -7 - 3t$ (1) | | |
| | | $7 - 3s = 1$ (2) | | |
| | | $-4 + s = 8 + 2t$ (3) | M1 | |
| | | (2) $\Rightarrow\ s = 2$, sub. (1) $\Rightarrow\ t = -5$ | B1 M1 | |
| | | check (3) $-4 + 2 = 8 - 10$, true $\therefore$ intersect | A1 | |
| | | intersect at $(7\mathbf{j} - 4\mathbf{k}) + 2(4\mathbf{i} - 3\mathbf{j} + \mathbf{k}) = (8\mathbf{i} + \mathbf{j} - 2\mathbf{k})$ | A1 | |
| | *(b)* | $= \cos^{-1}\left\lvert \frac{4\times(-3) + (-3)\times 0 + 1\times 2}{\sqrt{16+9+1}\times\sqrt{9+0+4}} \right\rvert$ | M1 A1 | |
| | | $= \cos^{-1}\left\lvert \frac{-10}{\sqrt{26}\times\sqrt{13}} \right\rvert = 57.0°$ (1dp) | M1 A1 | **(9)** |

| | | | |
|---|---|---|---|
| **5.** *(a)* | $\frac{dx}{dt} = \frac{1\times(2-t)-t\times(-1)}{(2-t)^2} = \frac{2}{(2-t)^2}$, $\quad \frac{dy}{dt} = -(1+t)^{-2}$ | M1 B1 | |
| | $\frac{dy}{dx} = -\frac{1}{(1+t)^2} \div \frac{2}{(2-t)^2} = -\frac{(2-t)^2}{2(1+t)^2} = -\frac{1}{2}\left(\frac{2-t}{1+t}\right)^2$ | M1 A1 | |
| *(b)* | $t = 1,\ x = 1,\ y = \frac{1}{2}$, grad $= -\frac{1}{8}$ | B1 | |
| | grad of normal = 8 | | |
| | $\therefore\ y - \frac{1}{2} = 8(x-1) \quad [\,y = 8x - \frac{15}{2}\,]$ | M1 A1 | |
| *(c)* | $x(2-t) = t$ | M1 | |
| | $2x = t(1+x),\ t = \frac{2x}{1+x}$ | A1 | |
| | $y = \frac{1}{1+\frac{2x}{1+x}} = \frac{1+x}{(1+x)+2x} \quad \therefore\ y = \frac{1+x}{1+3x}$ | M1 A1 | **(11)** |
| **6.** *(a)* | $= \int (\sec^2 x - 1)\ dx$ | M1 | |
| | $= \tan x - x + c$ | M1 A1 | |
| *(b)* | $= \int \frac{\sin x}{\cos x}\ dx$, let $u = \cos x$, $\frac{du}{dx} = -\sin x$ | M1 | |
| | $= \int \frac{1}{u} \times (-1)\ du = -\int \frac{1}{u}\ du$ | A1 | |
| | $= -\ln\lvert u\rvert + c = \ln\lvert u^{-1}\rvert + c = \ln\lvert\sec x\rvert + c$ | M1 A1 | |
| *(c)* | volume $= \pi\int_0^{\frac{\pi}{3}} x\tan^2 x\ dx$ | M1 | |
| | $u = x,\ u' = 1,\ v' = \tan^2 x,\ v = \tan x - x$ | M1 | |
| | $I = x(\tan x - x) - \int (\tan x - x)\ dx$ | A1 | |
| | $= x\tan x - x^2 - \ln\lvert\sec x\rvert + \frac{1}{2}x^2 + c$ | A1 | |
| | volume $= \pi[x\tan x - \frac{1}{2}x^2 - \ln\lvert\sec x\rvert]_0^{\frac{\pi}{3}}$ | | |
| | $= \pi\{(\frac{1}{3}\sqrt{3}\,\pi - \frac{1}{18}\pi^2 - \ln 2) - (0)\} = \frac{1}{18}\pi^2(6\sqrt{3} - \pi) - \pi\ln 2$ | M1 A1 | **(13)** |
| **7.** *(a)* | $\frac{dV}{dt} = -kV,\quad \frac{dV}{dh} = 10\pi h - \pi h^2$ | B2 | |
| | $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt} \quad \therefore\ -kV = (10\pi h - \pi h^2)\frac{dh}{dt}$ | M1 | |
| | $-\frac{1}{3}k\pi h^2(15-h) = \pi h(10-h)\frac{dh}{dt}$ | | |
| | $-kh(15-h) = 3(10-h)\frac{dh}{dt} \quad \therefore\ \frac{dh}{dt} = -\frac{kh(15-h)}{3(10-h)}$ | M1 A1 | |
| *(b)* | $\frac{3(10-h)}{h(15-h)} \equiv \frac{A}{h} + \frac{B}{15-h}$, $\quad 3(10-h) \equiv A(15-h) + Bh$ | M1 | |
| | $h = 0 \Rightarrow A = 2,\ h = 15 \Rightarrow B = -1 \quad \therefore\ \frac{3(10-h)}{h(15-h)} \equiv \frac{2}{h} - \frac{1}{15-h}$ | A2 | |
| *(c)* | $\int \frac{3(10-h)}{h(15-h)}\ dh = \int -k\ dt, \quad \int (\frac{2}{h} - \frac{1}{15-h})\ dh = \int -k\ dt$ | M1 | |
| | $2\ln\lvert h\rvert + \ln\lvert 15-h\rvert = -kt + c$ | M1 A1 | |
| | $t = 0,\ h = 5 \Rightarrow\ 2\ln 5 + \ln 10 = c,\quad c = \ln 250$ | M1 | |
| | $2\ln\lvert h\rvert + \ln\lvert 15-h\rvert - \ln 250 = -kt$ | | |
| | $\ln\frac{h^2(15-h)}{250} = -kt,\quad \frac{h^2(15-h)}{250} = e^{-kt},\quad h^2(15-h) = 250e^{-kt}$ | M1 A1 | |
| *(d)* | $t = 2,\ h = 4 \Rightarrow\ 176 = 250e^{-2k}$ | M1 | |
| | $k = -\frac{1}{2}\ln\frac{176}{250} = 0.175$ (3sf) | M1 A1 | **(17)** |
| | | Total | **(75)** |